Recall $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$, for $Re(s) > 1$.

We've already seen that $\zeta(s)$ admits analytic continuation to $Re(s) > 0$. Today we prove meromorphic continuation to $s \in \mathbb{C}$ and functional equation.

<u>Theorem</u>: The function $\zeta(s)$ admits meromorphic continuation to the entire complex plane and verifies the functional equation

$$\xi(s) := \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s) = \xi(1-s).$$

The only pole of $\zeta(s)$ is at $s=1$, it is simple with residue $\operatorname{res}_{s=1} \zeta(s) = 1$.

<u>Motivation for proof</u>:

We note that for $Re(s) > 1$, we have

$$\xi(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \sum_{n\geq 1} (\pi n^2)^{-s/2} \Gamma\left(\frac{s}{2}\right)$$

$$= \sum_{n\geq 1} (\pi n^2)^{-s/2} \mathcal{M}(e^{-y})\left(\frac{s}{2}\right)$$

$$= \sum_{n\geq 1} \mathcal{M}(e^{-\pi n^2 y})\left(\frac{s}{2}\right)$$

$$= \mathcal{M}\Big(\underbrace{\sum_{n\geq 1} e^{-\pi n^2 y}}_{\omega(y)}\Big)\left(\frac{s}{2}\right).$$

$$= \mathcal{M}(\omega(y))\left(\frac{s}{2}\right).$$

Define $\theta : (0,\infty) \to \mathbb{R}$ given by

$$\theta(x) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x}.$$

It converges absolutely for $x>0$:

$$|\theta(x)| \leq 1 + \int_{-\infty}^{\infty} e^{-\pi y^2 x}\, dy \leq 1 + \frac{1}{\sqrt{x}}.$$

Also, for $x \geq 1$:

$$\sum_{n=1}^{\infty} e^{-\pi n^2 x} = e^{-\pi x} \sum_{n\geq 1} e^{-\pi(n^2-1)x} \leq e^{-\pi x} \sum_{n\geq 1} e^{-\pi(n^2-1)}$$

$$\ll e^{-\pi x}.$$

Therefore $\theta(x) = 1 + O(e^{-\pi x})$, for $x \geq 1$.

Hence $\omega(x) = \dfrac{\theta(x)-1}{2} \ll e^{-\pi x}$, for $x \geq 1$.

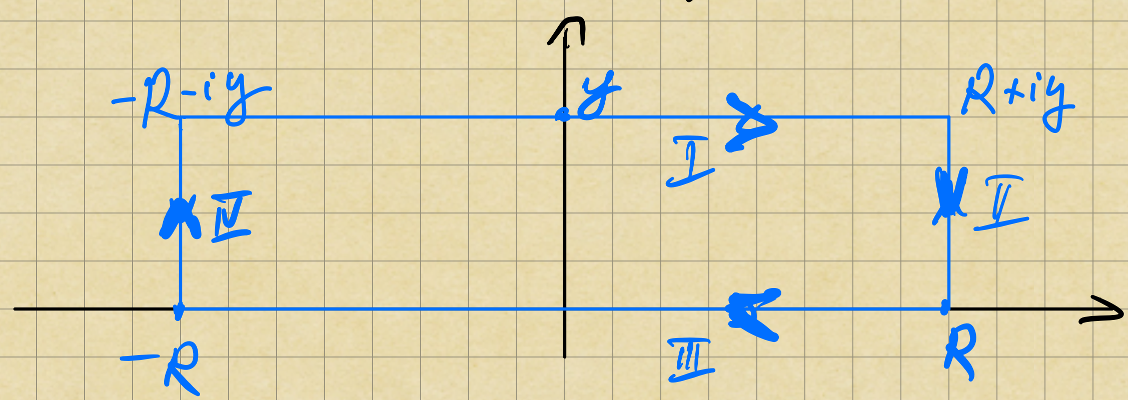
<u>Lemma</u>: Let $f(x) = e^{-\pi x^2} \in \mathcal{S}(\mathbb{R})$.

Then $\hat{f}(y) = e^{-\pi y^2} = f(y)$.

Proof: By completing the square,

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x y} = e^{-\pi y^2} \int_{-\infty}^{\infty} e^{-\pi (x+iy)^2} dx$$

$f(z) = e^{-\pi z^2}$ is a holomorphic function for $z \in \mathbb{C}$



From Cauchy's theorem:

$$\int_{-R+iy}^{R+iy} f(z)dz + \int_{R+iy}^{R} f(z)dz + \int_{R}^{-R} f(z)dz + \int_{-R}^{-R+iy} f(z)dz = 0.$$

Note that $|f(z)| = e^{-\pi(\mathrm{Re}(z)^2 - \mathrm{Im}(z)^2)}$.

Hence the second and fourth integral tend to 0 as $R \to \infty$.

$$\int_{-\infty}^{\infty} e^{-\pi (x+iy)^2} dx = \lim_{R\to\infty} \int_{-R+iy}^{R+iy} f(z)dz = -\lim_{R\to\infty} \int_{R}^{-R} f(z)dz$$

$$= \int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1. \qquad \square$$

Theorem (Functional equation for $\theta(x)$)

For all $x>0$, we have $\theta(x) = \frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right)$.

Proof: We've just seen $f(x) = e^{-\pi x^2} \in S(\mathbb{R})$

and $\hat{f}(x) = f(x)$.

Therefore, for $c>0$, let

$$f_c(x) = e^{-\pi c x^2} = f(\sqrt{c}\,x) \in S(\mathbb{R})$$

Hence $\hat{f_c}(y) = \frac{1}{\sqrt{c}}\,\hat{f}\left(\frac{y}{\sqrt{c}}\right) = \frac{1}{\sqrt{c}}\,e^{-\frac{\pi y^2}{c}}$.

From Poisson summation, we see

$$\theta(x) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 x} = \sum_{n\in\mathbb{Z}} f_x(n) = \sum_{n\in\mathbb{Z}} \hat{f_x}(n)$$

$$= \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{x}}\,e^{-\frac{\pi n^2}{x}} = \frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right). \quad \square$$

Proof of main theorem:

Denote $w(x) = \sum_{n\geq 1} e^{-\pi n^2 x} = \frac{\theta(x)-1}{2}$.

Then we have $w\left(\frac{1}{x}\right) = \sqrt{x}\,w(x) + \frac{\sqrt{x}-1}{2}$.

(from functional equation of $\theta(x)$).

Hence, for $Re(s)>1$, we have

$$\xi(s) = \pi^{-s/2}\,\Gamma\!\left(\tfrac{s}{2}\right)\zeta(s) = \mathcal{M}(\omega(x))\!\left(\tfrac{s}{2}\right)$$

$$= \int_0^\infty \omega(x)\, x^{s/2}\,\frac{dx}{x}$$

$$= \int_1^\infty \omega(x)\, x^{s/2}\,\frac{dx}{x} + \int_1^\infty \omega\!\left(\tfrac{1}{x}\right) x^{-\frac{s}{2}}\,\frac{dx}{x}$$

$$= \int_1^\infty \omega(x)\, x^{s/2}\,\frac{dx}{x} + \int_1^\infty \left(\sqrt{x}\,\omega(x) + \frac{\sqrt{x}-1}{2}\right) x^{-\frac{s}{2}}\,\frac{dx}{x}$$

$$= -\frac{1}{s} + \frac{1}{s-1} + \int_1^\infty \omega(x)\left(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\right)\frac{dx}{x}. \quad \circledast$$

We note that the integral is absolutely and uniformly convergent for all $s \in \mathbb{C}$, (since $\omega(x) \ll e^{-\pi x}$), hence it defines a holomorphic function.

In $\circledast$ we can interchange $s \leftrightarrow 1-s$, so the functional equation follows.

The only poles of $\xi(s)$ are at 0 and 1, and they are simple.

There are no zeros of $\xi(s)$ in $\mathrm{Re}(s) > 1$

(Since $\Gamma(s)$ has no zeros and $\zeta(s)$ has no zeros in $\mathrm{Re}(s) > 1$)

Therefore by functional equation, $\xi(s)$ has no zeros in $Re(s) < 0$.

Corollary (Zeros and poles of $\zeta(s)$)

$\zeta(s)$ is a meromorphic function with

- A simple pole at $s=1$ and no other poles.
- 'Trivial zeros' at $s = -2, -4, -6, \ldots$ and no other zeros in $Re(s) < 0$.
- 'Non-trivial' zeros ρ with $Re(\rho) \in [0, 1]$.
- No zeros with $Re(s) > 1$.

Proof: We have that $\zeta(s) = \dfrac{\pi^{s/2}\, \Gamma\left(\frac{1-s}{2}\right)}{\pi^{\frac{(1-s)}{2}}\, \Gamma\left(\frac{s}{2}\right)} \zeta(1-s)$.

$\pi^{s/2}$ and $\pi^{\frac{1-s}{2}}$ have no poles or zeros in complex plane
$\Gamma\left(\frac{s}{2}\right)$ has simple poles at $s = 0, -2, -4, \ldots$ and no zeros.

Conclusion follows. $\square$

The location of the zeros of a complex variable gives us a lot of information. We'll see later some important consequences for $\zeta(s)$ and the distribution of primes. We begin by reviewing some facts from complex analysis.

<u>Lemma</u> (Jensen's inequality)
(Size of analytic function controls density of zeros)

Let $f(z)$ an analytic function on the disk $|z| \le R$. If $f(0) \neq 0$, the number of zeros of f in the disk $|z| < R/2$ is bounded by

$$2 \max_{|z|=R} \log \left| \frac{f(z)}{f(0)} \right| .$$

<u>Proof</u>: Let $z_1, \ldots, z_k$ denote the zeros of f in the disk $|z| < R/2$.

Note that if $|z| = R$, then

$$|z - z'| = |\bar{z} - \bar{z}'| = \left| \frac{z\bar{z} - z\bar{z}'}{z} \right| = \left| \frac{R^2 - z\bar{z}'}{R} \right| .$$

Let $h(z) = f(z) \prod_{j=1}^{k} \left(\frac{R^2 - z\bar{z}_j}{R(z - z_j)} \right)$.

Then $h(z)$ analytic on $|z| \le R$ and $|h(z)| = |f(z)|$ if $|z| = R$.

By maximum modulus principle,

$$\max_{|z|=R} |f(z)| = \max_{|z|=R} |h(z)| \geq |h(0)| = |f(0)| \prod_{j=1}^{K} \frac{R}{|z_j|} \geq |f(0)| 2^K.$$

Therefore $K \leq \frac{1}{\log 2} \max_{|z|=R} \log \left| \frac{f(z)}{f(0)} \right|$. $\square$

We denote $N(T)$ the number of non-trivial zeros of $\zeta(s)$ with $|\mathrm{Im}(\rho)| \leq T$.

Lemma (Zeros of $\zeta(s)$ are not too dense)

For $T \geq 2$, we have $N(T) - N(T-1) \ll \log T$

and $N(T) \ll T \log T$.

Proof: We know that for $\mathrm{Re}(s) > 0$,

$$\zeta(s) = \frac{1}{s-1} + 1 - s \int_1^\infty \frac{\{y\}}{y^{s+1}} dy.$$

This implies that for $\sigma \geq \frac{1}{2}$ and $|t| \geq 1$,

we have $|\zeta(\sigma + it)| \ll |t|$.

We can integrate by parts, and obtain

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s(s+1) \int_1^\infty \frac{\{y\}^2 - \{y\}}{2y^{s+2}} dy.$$

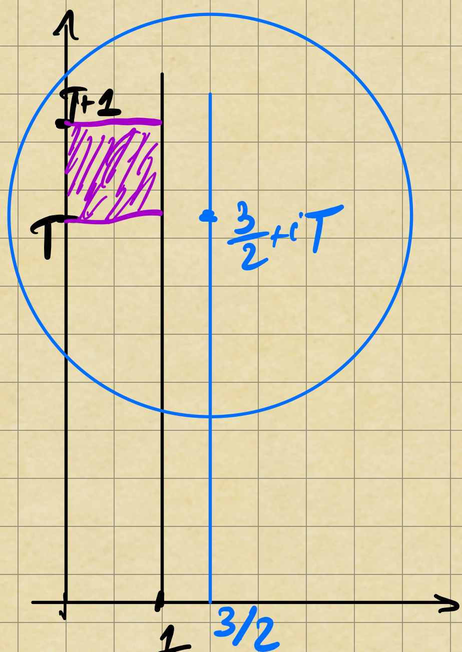
This is well defined for $\Re(s) > -1$ and it implies that if $\sigma \geq -\frac{1}{2}$, $|t| \geq 1$,

$$|\zeta(\sigma+it)| \ll (1+|t|)^2.$$

We can iterate the process of integration by parts, and for any $N \in \mathbb{N}$, we have that if $\sigma > -N+\frac{1}{2}$, $|t| \geq 1$

$$|\zeta(\sigma+it)| \ll_N (1+|t|)^{N+1}.$$

Suppose $T \geq 10$ (conclusion follows otherwise since there are finitely many non-trivial zeros with $|\operatorname{Im} \rho| \leq 10$).



Let $g(z) = \zeta(\frac{3}{2}+z+iT)$. Then $g(z)$ analytic in $|z| \leq 4$ and it satisfies $|g(z)| \ll T^4$ in this disk.

Also $|g(0)| = |\zeta(\frac{3}{2}+iT)| \neq 0$.

and $\frac{1}{|g(0)|} = \left|\frac{1}{\zeta(\frac{3}{2}+iT)}\right|$

$$= \left|\zeta_\mu\left(\frac{3}{2}+iT\right)\right| = \left|\sum_{n\geq 1} \frac{\mu(n)}{n^{3/2+iT}}\right| \leq \sum_{n\geq 1} \frac{1}{n^{3/2}} = \zeta\left(\frac{3}{2}\right).$$

From Jensen's inequality, $g(z)$ can have $O(\log T)$ zeros with $|z| \le 2$. This implies $\xi(s)$ has at most $O(\log T)$ zeros with $T \le \operatorname{Im} s \le T+1$.

(Since $\{0 \le \operatorname{Re}(s) \le 1,\ T \le \operatorname{Im}(s) \le T+1\} \subset \{s : |s - \frac{3}{2} - iT| \le 2\}$).

$\square$